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ORIGINAL PAPER

Self-similar solutions for the nonlinear dispersion of a chemical pollutant into a river flow

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Abstract We study the nonlinear coupled boundary value problem arising from the nonlinear dispersion of a chemical pollutant initially in a river. We convert the governing system of partial differential equations into a system of ordinary differential equations through a similarity transformation, which has not previously been done for these problems. The similarity solutions can then be obtained from the resulting boundary value problem. Both problems of uniform and variable initial profiles are considered, and exact solution profiles are obtained. We next generalize these results to account for the case where a non-zero concentration of the pollutant can be added for positive values of time. Physically, this models the diffusion of a chemical pollutant into a river when a spill is in progress. Since solutions under this model are governed by more complicated equations, numerical solutions are obtained. Finally, we consider solutions which exhibit wave-like structures. Such solutions model the propagation of waves in a river under the presence of a pollutant. A number of interesting physical observations are made during the analysis of these solutions.

Keywords Pollutant dispersion · Similarity solutions · Nonlinear partial differential equations

1 Introduction

The development of accurate methods and models to predict the spread of a pollutant once a discharge has been detected is an area of vital research [1]. Spread of pollutants in a fluid flow depends largely on concentration coefficients [2], which may

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be determined empirically for a specific type of pollutant. Investigations such as [3] and the current one can help in identifying the pollutant physical properties (and the related mathematical parameters) likely to cause the greatest impact in the spread and concentration of such a pollutant. Such investigations, as well as the complementary experimental works, in, say, large scale water treatment and redistribution net-works, are of great relevance [4–9]. Numerical results for a pollutant ejected into a channel flow were given in [10], whereas some analytical results for the same problem were obtained in [11].

In the present paper, we study the nonlinear coupled boundary value problem arising from the nonlinear dispersion of a chemical pollutant initially in a river. This system consists of two partial differential equations: one for the velocity of the river flow, and the other for the concentration of the pollutant. A formulation of this problem is given in Sect. 2. Then, in Sect. 3, we convert the governing system of partial differential equations into a system of ordinary differential equations through a similarity transformation. The similarity solutions can then be obtained from the resulting boundary value problem. Both the problems of uniform and variable initial profiles are considered, and exact solution profiles are obtained. These results differ from those in the literature, as self-similar structures have not been considered for this model previously (see [10,11]). However, as we shall demonstrate, these self-similar structures allow us to study some interesting aspects of the chemical pollution problem.

We next generalize results for the flow and concentration variables to account for the case where a non-zero concentration of the chemical pollutant can be added for positive values of time. A derivation of this model is given in Sect. 4. Physically, this model allows us to study the diffusion of a pollutant into a river when a spill is in progress. Since solutions under this model are governed by complicated equations, numerical solutions are obtained. We discuss the qualitative behavior of these similarity solutions with changes in various model parameters. Again, the results here are possible since we consider self-similar solutions to the problem.

In Sect. 5, we consider solutions which exhibit wave-like structures. Such solutions model the propagation of waves in a river under the presence of a pollutant. A number of interesting physical observations are made during the analysis of these solutions. Through a simple asymptotic analysis, we are able to determine that the chemical pollutant concentration should tend toward some constant value in the direction of wave prorogation.

The present results constitute an interesting study of a pollution problem, and hence the results should be of relevance to those working in the areas of environmental engineering, as well as applied mathematics and fluid dynamics related to the introduction of a chemical to a fluid flow. While the results for the diffusion of a pollutant already present at time $t = 0$ are standard, the solutions for the case where a pollutant spill is still in progress are very complicated to find (owing to the added complexity inherent in the complicated model). In Sect. 6, we summarize our findings and provide several conclusions.

2 Formulation of the problem

We consider the problem of the dispersion of a pollutant in a river flow previously discussed by Makinde and Moitsheki [8]. The assumptions are as follows:

(i) Initially, the flow in the river is incompressible, fully developed with a constant viscosity and that a given pollutant is injected into the river. The fluid viscosity then changes due to the concentration of the pollutant, i.e. the fluid dynamic viscosity is now pollutant concentration dependent.
(ii) There are no source-terms, so new pollutants are not added. Hence, this model addresses the dispersion of existing pollutants in the river.

Under these flow conditions the problem is reduced mathematically to a transient coupled fluid flow and mass transfer problem [8] as

$$\rho \frac{\partial u}{\partial \bar{t}} = -\frac{\partial \bar{P}}{\partial \bar{x}} + \frac{\partial}{\partial \bar{y}} \left(\bar{\mu} \left(C \right) \frac{\partial u}{\partial \bar{y}} \right), \tag{2.1}$$

$$\frac{\partial C}{\partial \bar{t}} = \frac{\partial}{\partial \bar{y}} \left(\bar{D} \left(C \right) \frac{\partial C}{\partial \bar{y}} \right), \tag{2.2}$$

subject to the initial and boundary conditions

$$u\left(\bar{y}, 0\right) = U\left(y\right), \; C\left(\bar{y}, 0\right) = C_0, \tag{2.3}$$

$$\frac{\partial u}{\partial \bar{y}}\left(0, \bar{t}\right) = 0, \; \frac{\partial C}{\partial \bar{y}}\left(0, \bar{t}\right) = 0, \; \text{for } \bar{t} > 0. \tag{2.4}$$

Here, u is the axial velocity of the fluid, C is the chemical pollutant concentration, C_0 is the chemical pollutant reference concentration, C_w is the chemical pollutant concentration at the walls, S is the pollutant external source function, g is the gravitational acceleration, ρ is the density, β is the concentration expansion coefficient, a is the channel half width, and $\bar{P}$ is the fluid pressure. The function $U\left(y\right)$ is the initial fluid velocity profile, which will take specific forms as needed. A uniform flow corresponds to $U\left(y\right) = U_0$, a constant.

The pollutant concentration dependent fluid dynamic viscosity $\bar{\mu}$ and mass diffusivity may be assumed to take the form of exponential functions (see [10,11]) as

$$\bar{\mu} = \mu_0 \exp\left(b_1 \left(C - C_0\right)/C_0\right), \text{ and } \bar{D} = D_0 \exp\left(b_2 \left(C - C_0\right)/C_0\right), \tag{2.5}$$

where μ_0 is the viscosity coefficient, D_0 is the mass diffusivity coefficient, b_1 is the viscosity variation parameter and b_2 is the mass diffusivity variation parameter.

In order to work with a non-dimensional form of the coupled boundary value problem (2.1)–(2.5), we introduce the following variables and constants:

$$y=\sqrt{\mathrm{Re}}\frac{\bar{y}}{a},\ x=\frac{\bar{x}}{a},\ w=\frac{a}{\nu}u,\ \phi=\frac{C}{C_0},\ t=\frac{\nu}{a^2}\bar{t},\ D=\frac{\bar{D}}{D_0},\ \mu=\frac{\bar{\mu}}{\mu_0},$$
$$P=\frac{a^2\bar{P}}{\rho\nu^2},\ K=-\mathrm{Re}^{-1}\frac{\partial P}{\partial x},\ \nu=\frac{\mu_0}{\rho},\ \alpha=b_1C_0,\ \beta=b_2C_0,\ \mathrm{Sc}=\frac{\nu}{D_0},$$
$$\sigma=\frac{\mathrm{Sc}}{2\mathrm{Re}}. \tag{2.6}$$

Substituting (2.7) into (2.1)–(2.6), we obtain

$$\frac{\partial w}{\partial t}=K+\frac{\partial}{\partial y}\left(\exp(\alpha\phi)\frac{\partial w}{\partial y}\right), \tag{2.7}$$
$$\frac{\partial \phi}{\partial t}=\frac{1}{2\sigma}\frac{\partial}{\partial y}\left(\exp(\beta\phi)\frac{\partial \phi}{\partial y}\right), \tag{2.8}$$
$$w(y,0)=W(y),\ \phi(y,0)=1, \tag{2.9}$$
$$\frac{\partial w}{\partial y}(0,t)=0,\ \frac{\partial \phi}{\partial y}(0,t)=0,\ \text{for}\ t>0. \tag{2.10}$$

Here K is the axial pressure gradient parameter, Sc is the Schmidt number, Re is the Reynolds number, α is the viscosity variation parameter, and β is the mass diffusivity variation parameter. We shall consider two distinct boundary conditions for the initial velocity profile: either $W(y)=w_0$ or $W(y)=w_0 y$ where w_0 is a constant. The first of these constitutes a constant initial flow, whereas the second gives a variable flow profile at the initial time.

3 Self-similar solutions

We shall now reduce (2.7)–(2.10) to a system of ordinary boundary value problems through an appropriate similarity transform. For both of the problems considered here, note that $\sigma>0$ and $\lambda>0$ are model parameters.

3.1 The case of a uniform initial flow

First, when we have a uniform initial flow, $W(y)=w_0$, we have the similarity transformation

$$w(y,t)=f(\eta)+Kt,\ \phi(y,t)=\frac{1}{\beta}\ln g(\eta)\ \text{where}\ \eta=\frac{y}{\sqrt{t}}. \tag{3.1}$$

With this transform, (2.7)–(2.10) reduce to a boundary value problem on a semi-infinite domain:

$$\frac{\eta}{2}f'+\left(f''+\lambda\frac{g'}{g}f'\right)g^{\lambda}=0, \tag{3.2}$$
$$gg''+\sigma\eta g'=0, \tag{3.3}$$

$$f'(0) = 0,\ \ g'(0) = 0, \tag{3.4}$$

$$g \to e^{\beta},\ \ f \to w_0 \text{ as } \eta \to \infty, \tag{3.5}$$

where $\lambda = \alpha/\beta > 0$ is a parameter. The first two boundary conditions follow directly from (2.10). The infinite boundary conditions come from the fact that

$$w_0 = w(y, 0) = \lim_{t \to 0^+} f\left(\frac{y}{\sqrt{t}}\right) = \lim_{\eta \to \infty} f(\eta), \tag{3.6}$$

while

$$1 = \phi(y, 0) = \frac{1}{\beta} \lim_{t \to 0^+} \ln g\left(\frac{y}{\sqrt{t}}\right) = \frac{1}{\beta} \lim_{\eta \to \infty} \ln g(\eta) \Rightarrow \lim_{\eta \to \infty} g(\eta) = e^{\beta}. \tag{3.7}$$

Observe that, due to the form of the boundary conditions, we can actually obtain an exact solution for the present case. If $g(\eta) = e^{\beta}$, then both boundary conditions for g are satisfied, while Eq. (3.3) is also satisfied. The equation for f then becomes $\eta f' + 2e^{\alpha} f'' = 0$. This equation, as well as the two boundary conditions on f, are satisfied provided that $f(\eta) = w_0$. From this, we therefore find the exact solution

$$w(y, t) = w_0 + Kt \text{ and } \phi(y, t) = 1. \tag{3.8}$$

In this case, the flow gradually increases or decreases in time from the initial uniform flow, depending on the sign of the axial pressure gradient parameter. Note that the flow remains uniform, in that it does not depend on the space coordinate, y. The concentration of the pollutant remains constant in this case.

3.2 The case of variable initial flow

Meanwhile, for the problem of a variable initial flow with $W(y) = w_0 y$, we introduce the different similarity transform

$$w(y, t) = \sqrt{t} f(\eta) + Kt,\ \ \phi(y, t) = \frac{1}{\beta} \ln g(\eta) \ \text{ where } \eta = \frac{y}{\sqrt{t}}. \tag{3.9}$$

Using (3.9), we find (2.7)–(2.10) become

$$\frac{\eta}{2} f' - \frac{1}{2} f + \left(f'' + \lambda \frac{g'}{g} f'\right) g^{\lambda} = 0, \tag{3.10}$$

$$gg'' + \sigma \eta g' = 0, \tag{3.11}$$

$$f'(0) = 0,\ \ g'(0) = 0, \tag{3.12}$$

$$g \to e^{\beta},\ \ f' \to w_0 \text{ as } \eta \to \infty. \tag{3.13}$$

The last boundary condition comes from the fact that

$$
\begin{aligned}
w_0 y = w(y, 0) &= \lim_{t \to 0^+} \sqrt{t} f\left(\frac{y}{\sqrt{t}}\right) = \lim_{t \to 0^+} \frac{f\left(y/\sqrt{t}\right)}{1/\sqrt{t}} = \lim_{t \to 0^+} y f'\left(y/\sqrt{t}\right) \\
&= y \lim_{\eta \to \infty} f'(\eta). \qquad (3.14)
\end{aligned}
$$

Due to the form of the boundary conditions, we can obtain an exact solution for this case as well. If $g(\eta) = e^{\beta}$, then both boundary conditions for g are satisfied, while Eq. (3.11) is also satisfied. The requirements on f then become

$$
\frac{\eta}{2} f' - \frac{1}{2} f + e^{\alpha} f'' = 0, \;\; f'(0) = 0, \;\; f' \to w_0 \text{ as } \eta \to \infty. \qquad (3.15)
$$

This equation is linear with variable coefficients. We find that the exact solution which takes into account the boundary conditions is

$$
f(\eta) = w_0 \left\{ \eta \operatorname{erf}\left(\frac{\eta}{2e^{\alpha/2}}\right) + \frac{2}{\sqrt{\pi}} \exp\left(\frac{\alpha}{2} - \frac{\eta^2}{4e^{\alpha}}\right) \right\}, \qquad (3.16)
$$

where erf denotes the error function

$$
\operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} \exp\left(-\tau^2\right) d\tau. \qquad (3.17)
$$

We therefore find meaningful solutions of the form

$$
\begin{aligned}
w(y, t) = w_0 &\left\{ y \operatorname{erf}\left(\frac{y}{2e^{\alpha/2}\sqrt{t}}\right) + \frac{2\sqrt{t}}{\sqrt{\pi}} \exp\left(\frac{\alpha}{2} - \frac{y^2}{4e^{\alpha} t}\right) \right\} \\
&+ Kt \text{ and } \phi(y, t) = 1. \qquad (3.18)
\end{aligned}
$$

In the present case, when the initial flow is variable in the spatial domain, the resulting flow is also variable in space. There is still a linear growth of the solution in time due to the axial pressure gradient. However, the pollutant concentration still remains constant (i.e., independent of space or time variables). In the following section, we shall study the effects of variable-time pollution concentrations, since such cases are more likely to model real-world spills of pollutants.

3.3 Remark on other non-admissible solutions

We note that the ordinary differential equation $gg'' + \sigma \eta g' = 0$ always has a second non-constant solution which is given by $g(\eta) = -\sigma \eta^2$. So, one may wonder why this is not the form of a solution. Note that this solution does not satisfy the required boundary conditions, nor does it satisfy the similarity transformation, since we must have $\phi(y, t) = \frac{1}{\beta} \ln g(\eta)$ (hence g must be positive) and $\sigma > 0$ (so the form of g we obtain is negative). Therefore, for the boundary conditions considered in this section, the only possible solution for g is the constant solution.

4 The inclusion of additional pollutants at positive time

The model given in (2.7)–(2.10) accounts for the diffusion of existing pollutants into a flow. However, it is possible that pollutants are added to the boundary of the system at positive time. Such a model would take the form

$$\frac{\partial w}{\partial t} = K + \frac{\partial}{\partial y}\left(\exp(\alpha\phi)\frac{\partial w}{\partial y}\right), \tag{4.1}$$

$$\frac{\partial \phi}{\partial t} = \frac{1}{2\sigma}\frac{\partial}{\partial y}\left(\exp(\beta\phi)\frac{\partial \phi}{\partial y}\right), \tag{4.2}$$

$$w(y,0) = W(y), \ \ \phi(y,0) = 1, \tag{4.3}$$

$$\frac{\partial w}{\partial y}(0,t) = 0, \ \ \frac{\partial \phi}{\partial y}(0,t) = \Phi(t), \ \text{for} \ \ t > 0. \tag{4.4}$$

Here the final boundary condition is modified so that the function $\Phi(t)$ gives additional pollutants at positive values of time. Based on the form of the solutions sought, it is natural to consider a function $\Phi(t)$ which decays as time gets large, so that most of the pollution occurs for small values of time. Let us consider $\Phi(t)$ that decays like

$$\Phi(t) \approx \frac{\delta}{\sqrt{t}} \tag{4.5}$$

where δ is a positive constant. Making use of the similarity transformation

$$\phi(y,t) = \frac{1}{\beta}\ln g(\eta) \ \text{ where } \eta = \frac{y}{\sqrt{t}}, \tag{4.6}$$

we see that we must have

$$\frac{\delta}{\sqrt{t}} = \frac{\partial \phi}{\partial y}(0,t) = \frac{1}{\beta}\lim_{y\to 0}\frac{\partial}{\partial y}\ln g\left(\frac{y}{\sqrt{t}}\right) = \frac{1}{\beta\sqrt{t}}\frac{g'(0)}{g(0)} \Rightarrow g'(0) - \beta\delta g(0) = 0. \tag{4.7}$$

This is a mixed condition for the function g.

4.1 The uniform initial flow with added pollutants

When we have a uniform initial flow, $W(y) = w_0$, we use the similarity transformation (3.1), which gives us

$$\frac{\eta}{2}f' + \left(f'' + \lambda\frac{g'}{g}f'\right)g^{\lambda} = 0, \tag{4.8}$$

$$gg'' + \sigma\eta g' = 0, \tag{4.9}$$

$$f'(0) = 0, \; g'(0) - \beta\delta g(0) = 0, \tag{4.10}$$

$$g \to e^{\beta}, \; f \to w_0 \text{ as } \eta \to \infty, \tag{4.11}$$

where $\lambda = \alpha/\beta > 0$ is a parameter.

As was true in Sect. 3.1, an exact solution for Eq. (4.8) is $f(\eta) = w_0$. Therefore, we have only to solve the relevant equations for $g(\eta)$. To better understand solutions $g(\eta)$ of Eq. (4.9), let us write (4.9) as

$$g'' + \sigma\eta\frac{g'}{g} = 0. \tag{4.12}$$

Integrating (4.12) once, we have

$$g'(\eta) - g'(0) + \sigma\int_0^{\eta} \tau\frac{g'(\tau)}{g(\tau)} = 0. \tag{4.13}$$

Integrating the last term in (4.13) by parts and using the condition (4.10) gives $g'(0) = \beta\delta g(0)$. Hence we obtain

$$g'(\eta) = \beta\delta g(0) - \sigma\eta\ln(g(\eta)) + \sigma\int_0^{\eta} \ln(g(\tau))\,d\tau. \tag{4.14}$$

Performing one more integration on (4.14), we obtain a nonlinear integral equation involving $g(\eta)$ as

$$g(\eta) = g(0) + \beta\delta g(0)\,\eta + \sigma\int_0^{\eta} (\eta - 2\tau)\ln(g(\tau))\,d\tau. \tag{4.15}$$

Taking the appropriate limit as $\eta \to \infty$, we see from (4.11) that the integral term should scale like

$$\int_0^{\eta} (\eta - 2\tau)\ln(g(\tau))\,d\tau \sim \frac{e^{\beta} - g(0)}{\sigma} - \frac{\beta\delta g(0)}{\sigma}\eta, \tag{4.16}$$

for very large η.

Equation (4.15) is strongly nonlinear, with the natural logarithm of $g(\eta)$ entering into the integral term. Therefore, we resort to numerical solutions of $g(\eta)$. In Fig. 1, we provide several plots of the numerical solution $g(\eta)$ to the boundary value problem (4.8)–(4.11). We see that an increase in the parameter σ results in an increase in the solution profile for small values of η while for large value of η there is no strong effect. An increase in the parameter β results in an increase in the solution profiles. Finally, an increase in the parameter δ results in a decrease in the solution profiles.

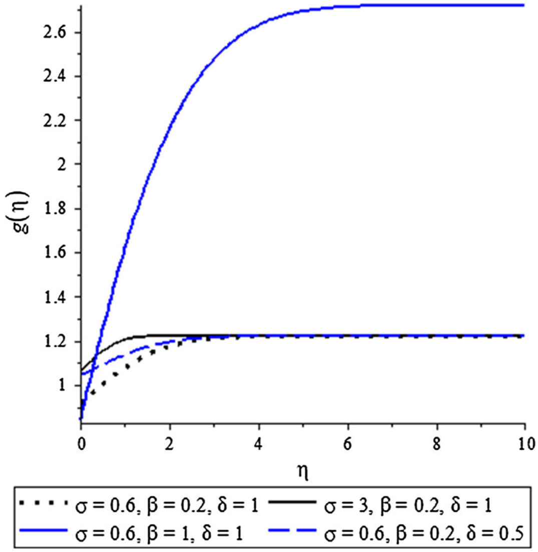

Fig. 1 Plot of the solution $g(\eta)$ to the boundary value problem (4.8)–(4.11). We see that an increase in the parameter σ results in an increase in the solution profile for small values of η while for large value of η there is no strong effect. An increase in the parameter β results in an increase in the solution profiles. Finally, an increase in the parameter δ results in a decrease in the solution profiles

We remark that all numerical computations are carried out in Maple 17 using the boundary value problem solver under the 'dsolve' command. We find that a numerical infinity value of $\eta^* = 10$ is sufficient to obtain accurate solutions.

4.2 The variable initial flow with added pollutants

For a variable initial flow with $W(y) = w_0 y$, we use the similarity transformation (3.8), finding

$$\frac{\eta}{2} f' - \frac{1}{2} f + \left(f'' + \lambda \frac{g'}{g} f' \right) g^{\lambda} = 0, \tag{4.17}$$

$$g g'' + \sigma \eta g' = 0, \tag{4.18}$$

$$f'(0) = 0, \ \ g'(0) - \beta \delta g(0) = 0, \tag{4.19}$$

$$g \to e^{\beta}, \ \ f' \to w_0 \text{ as } \eta \to \infty. \tag{4.20}$$

Note that solutions $f(\eta)$ will not be constant, owing to the boundary condition (4.20). Therefore, the solutions of (4.17)–(4.20) will be even more complicated than those in previous sections. As such, we solve the resulting boundary value problem numerically.

We give numerical solutions for $f(\eta)$ in Fig. 2, while we provide numerical solutions for $g(\eta)$ in Fig. 3. We find that an increase in the parameter w_0 results in a strong increase in the numerical solution for $f(\eta)$. If we increase the parameter λ, we find that there is an increase in the solution for $f(\eta)$. Meanwhile, an increase in the parameter β results in an increase in the solution $f(\eta)$ in the small η regime. However, an increase in β results in a large increase in the solution profiles for $g(\eta)$.

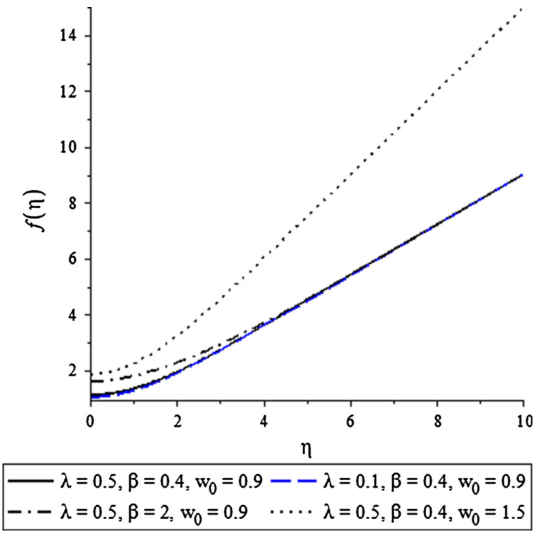

Fig. 2 Plot of the solution $f(\eta)$ to the boundary value problem (4.17)–(4.20). We see that an increase in the parameter w_0 results in a strong increase in the solutions. Next, an increase in the parameter λ results in a rather small increase in the solution profiles. Lastly, an increase in the parameter β results in an increase in the solution profiles, primarily for small values of η

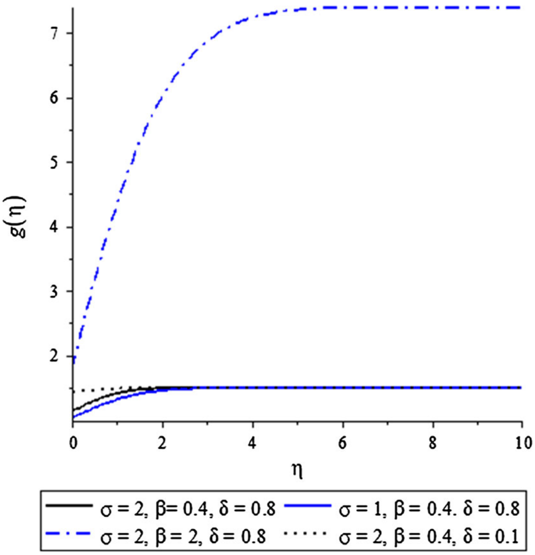

Fig. 3 Plot of the solution $g(\eta)$ to the boundary value problem (4.17)–(4.20). We see that an increase in the parameter σ results in an increase in the solution profile for small values of η while for large value of η there is no strong effect. Second, an increase in the parameter β results in a large increase in the solution profiles. Finally, an increase in the parameter δ results in a decrease in the solution profiles, primarily for small values of η

An increase in the parameter σ results in an increase in the numerical solution of $g(\eta)$ for small values of η, while for large values of η there is no strong effect. Finally, an increase in the parameter δ results in a decrease in the numerical solution profiles for $g(\eta)$, primarily for small values of η.

5 General non-similar solutions in the traveling wave case

Of course, self-similar solutions are one possibility for solutions which exhibit some degree of symmetry. Alternately, one may directly study the boundary value problem

(2.7)–(2.10) without such assumptions. One other possibility would be wave-like solutions. Such solutions possess a different form of symmetry, and would model waves propagating in a flow under the presence of a pollutant. To discuss such solutions, we make the assumption that $w(y,t) = \theta(z)\,, \phi(y,t) = \varphi(z)$ where z is the wave variable $z = x - ct$ and c denotes the wave speed. Equations (2.7)–(2.8) become

$$-c\theta' = K + \left(\exp(\alpha\varphi)\,\theta'\right)', \tag{5.1}$$

$$-c\varphi' = \frac{1}{2\sigma}\left(\exp(\beta\varphi)\,\varphi'\right)', \tag{5.2}$$

where a prime denotes differentiation with respect to z.

Note that (5.1) is essentially a linear equation for θ. Integrating (5.1), we have

$$m - c\theta = Kz + \exp(\alpha\varphi)\,\theta', \tag{5.3}$$

where m is a constant of integration. We have an inhomogeneous differential equation of first order for θ of the form

$$\theta'(z) + c\exp(-\alpha\varphi(z))\,\theta(z) = (m - Kz)\exp(-\alpha\varphi(z))\,. \tag{5.4}$$

The solution of (5.4) is then given in exact form by

$$\theta(z) = \theta(0)\exp\left(-c\int_0^z \exp(-\alpha\varphi(\tilde{z}))\,d\tilde{z}\right) + \int_0^z (m - K\tilde{z})\exp\left(-\alpha\varphi(\tilde{z}) - c\int_{\tilde{z}}^z \exp(-\alpha\varphi(\tau))\,d\tau\right)d\tilde{z}. \tag{5.5}$$

Here $\theta(0)$ is a free parameter. Therefore, once a solution φ to (5.2) is known, a solution to (5.1) can be given in exact form by the representation (5.5).

While (5.2) is rather simple to write, the equation is nonlinear for the unknown function φ and is hence challenging to solve. Integrating (5.2) with respect to z, we have

$$M - c\phi = \frac{1}{2\sigma}\exp(\beta\varphi)\,\varphi', \tag{5.6}$$

where M is a constant of integration. Rewriting (5.6) as

$$\varphi' = 2\sigma(M - c\varphi)\exp(-\beta\varphi)\,, \tag{5.7}$$

we may separate variables to obtain an implicit representation for ϕ in terms of z:

$$\frac{1}{2\sigma}\int_{\varphi(0)}^{\varphi(z)} \frac{\exp(\beta A)}{(M - cA)}dA = z. \tag{5.8}$$

The integral in (5.8) can be expressed as exponential integrals, which cannot in general be inverted (for arbitrary values of the model parameters). Therefore, a general exact solution is not forthcoming. Even though we cannot solve (5.7) exactly, we can obtain several results on the behavior of φ. First, let us note that the differential Eq. (5.7) has an equilibrium point $\varphi^* = M/c$. Note that

$$\frac{d}{d\varphi}\left\{2\sigma\left(M - c\varphi\right)\exp\left(-\beta\varphi\right)\right\}\Big|_{\varphi=\varphi^*} = -2\sigma c \exp\left(-\frac{\beta M}{c}\right). \tag{5.9}$$

By the linear stability test, the equilibrium $\varphi^* = M/c$ is stable provided that $c > 0$ and unstable when $c < 0$ (since σ is always positive). What this means is that a wave solution should tend toward an asymptotic value of M/c as $z \to \infty$ provided that the wave speed c is positive. Numerical solutions verify this type of behavior for the large z regime.

If β is a small parameter, say $|\beta| << 1$, then we can approximate (5.7) by

$$\varphi' = 2\sigma\left(M - c\varphi\right). \tag{5.10}$$

The exact solution to this equation is given by

$$\varphi\left(z\right) = \frac{M}{c} + \left(\varphi\left(0\right) - \frac{M}{c}\right)\exp\left(-2\sigma c z\right), \tag{5.11}$$

where $\varphi\left(0\right)$ is an arbitrary constant denoting the value of φ at $z = 0$. Note that this solution exhibits the asymptotic behavior predicted from the linear stability analysis. Then, in the small β regime we should have [from Eq. (5.5)] that

$$\begin{aligned}\theta\left(z\right) = \theta\left(0\right)\exp\left(-c\exp\left(-\alpha\frac{M}{c}\right)\int_0^z \exp\left(-\alpha\left(\varphi\left(0\right) - \frac{M}{c}\right)\exp\left(-2\sigma c\tilde{z}\right)\right)d\tilde{z}\right)\\ + e^{-\alpha M/c}\int_0^z \left(m - K\tilde{z}\right)\exp\left(-\alpha\left(\varphi\left(0\right) - \frac{M}{c}\right)e^{-2\sigma c\tilde{z}}\right.\\ \left. - c\, e^{-\alpha M/c}\int_{\tilde{z}}^z \exp\left(-\alpha\left(\varphi\left(0\right) - \frac{M}{c}\right)e^{-2\sigma c\tau}\right)d\tau\right)d\tilde{z}.\end{aligned} \tag{5.12}$$

Therefore, when β is small, Eqs. (5.11)–(5.12) give an approximate solution to the problem (5.1)–(5.2).

6 Conclusions

In the present paper, we have given similarity formulations for two models of the pollutant flow into a river. The first model assumes that all pollutants are dispersed by time $t = 0$, and that the pollutants disburse over time. The second model is a generalization of the first, which allows for pollutants to be added for positive values of time (and therefore pollution of the river can be ongoing). In the case where pollutants are all initially in the system at $t = 0$, we demonstrate that exact solutions are possible. Specific solutions for both uniform and variable initial flows are provided.

When additional pollutants are added for $t > 0$, the solution process is a bit more complicated, owing to the additional complexity of the problem. However, for both uniform and variable initial flows, it is possible to solve numerically the required boundary value problems in order to ascertain the behavior of solutions to the problem. Note that, for the case of a non-uniform initial flow (in particular, the flow which is linear in space at $t = 0$), we have the mixed boundary condition $g'(0) - \beta\delta g(0) = 0$, which also complicates the solution procedure. In the case of a uniform initial flow, we find that the flow remains constant, while the concentration function exhibits several interesting properties:

- An increase in the parameter σ results in an increase in the solution $g(\eta)$ for small values of η while for large value of η there is no strong effect;
- An increase in the parameter β results in an increase in the solution $g(\eta)$;
- An increase in the parameter δ results in a decrease in the solution $g(\eta)$.

However, in the case of non-uniform initial flows, we must solve a boundary problem for an unknown $f(\eta)$ and $g(\eta)$. In the case where the initial flow is a linear function of the space variable, we find that the velocity function $f(\eta)$ satisfies the following properties:

- An increase in the parameter w_0 results in a large increase in the solution $f(\eta)$.
- An increase in the parameter λ results in a small increase in the solution $f(\eta)$.
- An increase in the parameter β results in an increase in the solution $f(\eta)$ (particularly for small values of η).

Regarding the concentration function $g(\eta)$, we find that

- An increase in the parameter σ results in an increase in the solution $g(\eta)$ for small values of η;
- An increase in the parameter β results in a large increase in the solution $g(\eta)$;
- An increase in the parameter δ results in a decrease in the solution $g(\eta)$, primarily for small values of η.

Finally, we considered wave-like solutions to the river pollution model. These solutions model waves propagating in a flow under the presence of a pollutant. Such solutions take the form $w(y,t) = \theta(z)$ and $\phi(y,t) = \varphi(z)$ where z is the wave variable $z = x - ct$ and c denotes the wave speed. We were able to ascertain certain mathematical properties of these solutions. Whenever the wave speed is positive, the concentration profiles were shown to tend toward a positive equilibrium value in the $z \to \infty$ limit.

The present results constitute an analytical and numerical study of a pollution problem, and hence the results may be of relevance to those working in the areas of environmental engineering, as well as applied mathematics and fluid dynamics. In particular, for the case where pollutants are added into the river flow for $t > 0$, we see that the pollution concentration, $g(\eta)$, depends strongly on the model parameters. Therefore, if one is able to adequately measure all of the parameters that are used in the model (4.1)–(4.4), one should be able to ascertain the effects that the pollutant will have on the river by studying the solutions $g(\eta)$.

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